Journal of Applied Mechanics and Technical Physics, Vol. 49, No. 5, pp. 832–845, 2008

DIFFERENTIATION OF ENERGY FUNCTIONALS IN THE PROBLEM OF A CURVILINEAR CRACK IN A PLATE WITH A POSSIBLE CONTACT OF THE CRACK FACES

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UDC 539.375

A problem of equilibrium of a cracked plate is considered within the framework of the Kirchhoff–Love model. Non-penetration conditions in the form of inequalities (Signorini-type conditions) are set on the crack faces. The behavior of the energy functional is studied for the case of a rather smooth perturbation of the domain of the general form. Sufficient conditions for the existence of the energy functional derivative with respect to the parameter of domain perturbation are derived.

Key words: *elasticity, crack, Griffith criterion, variational inequality, energy functional derivative, non-smooth domain.*

Introduction. The Kirchhoff–Love model for a clamped plate is considered. The plate is in equilibrium under the action of an external force and contains a vertical curvilinear crack.

In this work, we study mathematical issues of the theory of cracks, which involves the Griffith fracture criterion. In accordance with this criterion, crack propagation starts at the moment when the derivative of the energy functional with respect to the domain perturbation parameter reaches a certain critical value depending only on the properties of the solid material [1, 2].

Various models that describe nonlinear cracks with one-sided restrictions were considered in [3–5].

The possibility of differentiating the energy functionals with respect to the domain perturbation parameter was considered by various authors. Examples of linear boundary-value problems in non-smooth domains can be found in [6–8]. Variations of solutions, stress-intensity coefficients, and other functions of geometric and mechanical parameters with variations of the crack shape or with the crack growth (including the cases with a curvilinear crack) were studied in [9–17].

The papers [18–28] deal with investigations of energy functional differentiability for boundary-value problems with one-sided restrictions on the boundary. Invariant integrals for various perturbations of the domain were obtained for such problems with the use of a formula for the derivative [19, 20, 25]. In [18–25], the cracks were assumed to be rectilinear or plane; otherwise, additional conditions were imposed on the domain perturbation. An asymptotic curve of the energy functional for curvilinear (surface) cracks was derived in [26–28].

In the present paper, we consider a general-type perturbation of a domain with a crack, which depends on a small parameter ε. The energy functional is determined in the perturbed domain. The behavior of the energy functional in the case of domain perturbation is studied by the method proposed in [26, 28]. In this case, in contrast to [19, 20, 22–24], a one-to-one correspondence between the sets of admissible displacements of the perturbed and non-perturbed problems is not needed. Sufficient conditions for the existence of the energy functional derivative are found.

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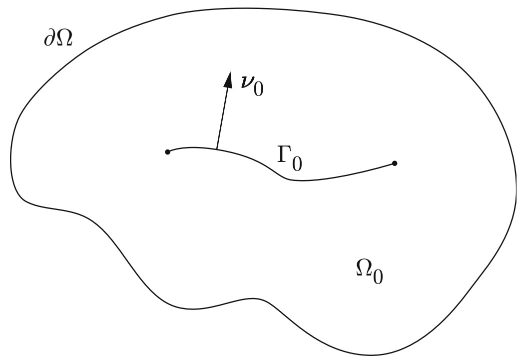

Fig. 1. Mid-surface of a plate with a crack.

1. Statement of the Problem and Formulation of the Basic Result. Let us consider a domain Ω with a smooth boundary $\partial\Omega$, which is bounded in a space $\mathbb{R}^2$. Let a curve Γ_0 be located rigorously inside the domain. Let us make some assumptions concerning Ω and Γ_0.

Assumption 1. Let the set $\{\Omega, \partial\Omega, \Gamma_0\}$ satisfy the following conditions: 1) the domain Ω can be divided into two subdomains Ω_1 and Ω_2 with a common boundary Γ; 2) the boundaries of the domains Ω_1 and Ω_2 are the Lipschitz boundaries; 3) $\overline{\Omega}_1 \cup \overline{\Omega}_2 = \overline{\Omega}$ and $\overline{\Omega}_1 \cap \overline{\Omega}_2 = \overline{\Gamma}$; 4) $\Gamma_0 \subset \Gamma$; 5) Γ_0 belongs to the class $C^{1,1}$ and is a regular curve; 6) Γ_0 can be extended to a smooth closed curve $\Sigma \subset \Omega$. Note, in accordance with condition 5, Γ_0 is a curve that does not intersect itself with a normal vector existing at each point of the curve (Fig. 1).

We assume that the crack Γ_0 in the space $\mathbb{R}^2$ is defined in the form

$$\Upsilon(\boldsymbol{x}) = 0, \qquad \boldsymbol{x} = (x_1, x_2) \in \Gamma_0, \tag{1}$$

where $\Upsilon \in C^{1,1}_{\mathrm{loc}}(\mathbb{R}^2)$.

Let us define the unit vector of the normal $\boldsymbol{\nu}_0 = (\nu_{01}, \nu_{02})$ to Γ_0. By virtue of Eq. (1) and smoothness of Υ, the components of the vector $\boldsymbol{\nu}_0$ have the form

$$\nu_{0i} = \frac{1}{|\nabla\Upsilon|}\frac{\partial\Upsilon}{\partial x_i}, \qquad i = 1, 2.$$

Note that $\nabla\Upsilon \neq 0$ for all $x \in \Gamma_0$ because Γ_0 is a regular surface. We assume that the chosen direction of the normal $\boldsymbol{\nu}_0$ defines the positive face Γ_0^+, and the direction $-\boldsymbol{\nu}_0$ indicates the negative face Γ_0^- of the crack Γ_0.

Let us define the domain $\Omega_0 = \Omega \setminus \Gamma_0$ corresponding to the mid-surface of the plate, which belongs to the plane Ox_1x_2 (Γ_0 is the trace of the crack on Ox_1x_2).

In what follows, the subscripts i, j, k, and l take the values from 1 to 2, except for cases indicated specially. Summation from 1 to 2 is performed over repeated subscripts.

Let us define the functional spaces

$$H^{1,0}(\Omega_0) = \{u \in H^1(\Omega_0)\colon \quad u = 0 \quad \text{almost everywhere on } \partial\Omega\},$$

$$H^{2,0}(\Omega_0) = \Big\{w \in H^2(\Omega_0)\colon \quad w = \frac{\partial w}{\partial \boldsymbol{n}} = 0 \quad \text{almost everywhere on } \partial\Omega\Big\},$$

where $\boldsymbol{n}$ is the outward unit normal to $\partial\Omega$, and introduce the notation $H(\Omega_0) = H^{1,0}(\Omega_0) \times H^{1,0}(\Omega_0) \times H^{2,0}(\Omega_0)$. It should be noted that the vector-functions from the space $H(\Omega_0)$ can take different values on the crack faces Γ_0^+ and Γ_0^-.

Let $\boldsymbol{\chi} = (\boldsymbol{W}, w)$ be a three-component vector-column of displacements of the points of the plate mid-surface, where $\boldsymbol{W} = (w_1, w_2)$ are the horizontal displacements and w are the vertical flexures. The formulas for the components of the strain tensor $\varepsilon_{ij}(\boldsymbol{W})$ and stress tensor $\sigma_{ij}(\boldsymbol{W})$ are written in dimensionless form as [3]

$$\varepsilon_{ij}(\boldsymbol{W}) = (1/2)(w_{i,j} + w_{j,i}),$$

$$\sigma_{11}(\boldsymbol{W}) = \varepsilon_{11}(\boldsymbol{W}) + k\varepsilon_{22}(\boldsymbol{W}), \qquad \sigma_{22}(\boldsymbol{W}) = \varepsilon_{22}(\boldsymbol{W}) + k\varepsilon_{11}(\boldsymbol{W}),$$

$$\sigma_{12}(\boldsymbol{W}) = (1-k)\varepsilon_{12}(\boldsymbol{W}), \qquad k = \text{const}, \qquad 0 < k < 1/2,$$

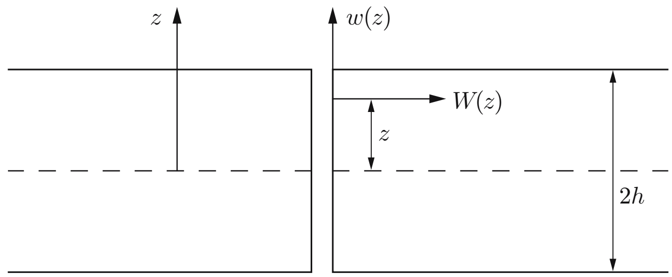

Fig. 2. Displacements of the plate points on the crack.

where k is Poisson's ratio; the subscripts after the comma indicate differentiation with respect to the corresponding coordinate.

We assume that the non-penetration condition is satisfied on the crack [3]:

$$[\boldsymbol{W}^{\mathrm{t}}]\nu_0 \geq |[(\nabla w)^{\mathrm{t}}\boldsymbol{\nu}_0]| \quad \text{almost everywhere on } \Gamma_0$$

($[u] = u|_{\Gamma_0^+} - u|_{\Gamma_0^-}$ is the jump of the function on the crack Γ_0).

Let $\boldsymbol{f} = (f_1, f_2, f_3)$ be a specified vector of external loading [$f_i \in C^1(\overline{\Omega})$, where $i = 1, 2, 3$]. We consider the energy functional of the plate

$$\Pi(\Omega_0; \boldsymbol{\chi}) = \frac{1}{2}\int\limits_{\Omega_0} \sigma_{ij}(\boldsymbol{W})\varepsilon_{ij}(\boldsymbol{W})\, d\Omega_0 + \frac{1}{2}\int\limits_{\Omega_0} b(w, w)\, d\Omega_0 - \int\limits_{\Omega_0} \boldsymbol{f}^{\mathrm{t}}\boldsymbol{\chi}\, d\Omega_0,$$

where the bilinear form $b(\,\cdot\,,\,\cdot\,)$ is defined as $b(\varphi, \psi) = \varphi_{,11}\psi_{,11} + \varphi_{,22}\psi_{,22} + k\varphi_{,11}\psi_{,22} + k\varphi_{,22}\psi_{,11} + 2(1-k)\varphi_{,12}\psi_{,12}$.

Then, we determine the set of admissible displacements of the points of the plate mid-surface:

$$K_0(\Omega_0) = \{\boldsymbol{\chi} \in H(\Omega_0)\colon \quad [\boldsymbol{W}^{\mathrm{t}}]\boldsymbol{\nu}_0 \geq |[(\nabla w)^{\mathrm{t}}\boldsymbol{\nu}_0]| \quad \text{almost everywhere on } \Gamma_0\}.$$

The problem of plate equilibrium can be formulated as the problem of minimization of the energy functional on the set of admissible displacements: we have to find a function $\boldsymbol{\chi}_0 \in K_0(\Omega_0)$ such that

$$\Pi(\Omega_0; \boldsymbol{\chi}_0) = \inf_{\boldsymbol{\chi} \in K_0(\Omega_0)} \Pi(\Omega_0; \boldsymbol{\chi}). \tag{2}$$

It is known that problem (2) has a unique solution [3] that satisfies the variational inequality

$$\int\limits_{\Omega_0} \sigma_{ij}(\boldsymbol{W}_0)\varepsilon_{ij}(\boldsymbol{W} - \boldsymbol{W}_0)\, d\Omega_0 + \int\limits_{\Omega_0} b(w_0, w - w_0)\, d\Omega_0 \geq \int\limits_{\Omega_0} \boldsymbol{f}^{\mathrm{t}}(\boldsymbol{\chi} - \boldsymbol{\chi}_0)\, d\Omega_0 \quad \forall\, \boldsymbol{\chi} \in K_0(\Omega_0). \tag{3}$$

The solution $\boldsymbol{\chi}_0$ of problem (2) determines the horizontal displacements and the vertical flexures of the plate mid-surface. In the Kirchhoff–Love model, the horizontal displacements $\boldsymbol{W}(z)$ of an arbitrary point (x_1, x_2, z) of the plate depend linearly on the distance from the plate mid-surface, whereas the vertical displacements $w(z)$ of this point coincide with the vertical deflections of the mid-surface [29]:

$$\boldsymbol{W}(z) = \boldsymbol{W}_0 - z\nabla w_0, \qquad w(z) = w_0, \qquad |z| \leq h$$

($2h$ is the plate thickness), as is shown in Fig. 2. In the present paper, we assume that $h = 1$.

For the small parameter $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$, we consider the perturbation $\boldsymbol{\Phi}_\varepsilon = (\Phi^1_\varepsilon(\boldsymbol{x}), \Phi^2_\varepsilon(\boldsymbol{x}))$, which is defined by smooth functions $\Phi^i \in C^1(-\varepsilon_0, \varepsilon_0; W^{2,\infty}(\mathbb{R}^2))$ and $\boldsymbol{\Phi}_0(\boldsymbol{x}) = \boldsymbol{x}$.

Let us fix ε and apply the coordinate transformation

$$\boldsymbol{y} = \boldsymbol{\Phi}_\varepsilon(\boldsymbol{x}) \tag{4}$$

for $\boldsymbol{x} \in \Omega_0$, $\boldsymbol{x} \in \partial\Omega$, and $\boldsymbol{x} \in \Gamma_0$. As a result, we obtain a perturbed domain $\Phi_\varepsilon(\Omega)$ and a perturbed crack $\Gamma_\varepsilon = \Phi_\varepsilon(\Gamma_0)$. We assume that the outer boundary of the domain remains unchanged, i.e., $\Phi_\varepsilon(\partial\Omega) = \partial\Omega$; moreover, for all admissible values of ε, the condition $\boldsymbol{n}(x) = \boldsymbol{n}^\varepsilon(\boldsymbol{\Phi}_\varepsilon(\boldsymbol{x}))$ almost everywhere on $\partial\Omega$ is satisfied ($\boldsymbol{n}^\varepsilon$ is the outward normal to $\partial\Omega$ in the new coordinates). This means that the outward unit normal $\boldsymbol{n}$ of the domain Ω transforms to the outward unit normal $\boldsymbol{n}^\varepsilon$ of the domain $\Phi_\varepsilon(\Omega)$. Let us define the perturbed domain with the crack as $\Omega_\varepsilon = \Phi_\varepsilon(\Omega) \setminus \overline{\Gamma}_\varepsilon$.

For the inverse transformation, we assume that $\boldsymbol{x} = \boldsymbol{\Phi}_\varepsilon^{-1}(\boldsymbol{y})$, where $\boldsymbol{\Phi}_\varepsilon^{-1} = (\Phi_{\varepsilon 1}^{-1}, \Phi_{\varepsilon 2}^{-1})$, and there are inclusions $\Phi_i^{-1} \in C^1(-\varepsilon_0, \varepsilon_0; W^{2,\infty}(\mathbb{R}^2))$.

Assumption 2. For all $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$, we assume that the set $\{\Phi_\varepsilon(\Omega), \Phi_\varepsilon(\partial\Omega), \Gamma_\varepsilon\}$ satisfies the conditions of Assumption 1.

Similar to the space $H(\Omega_0)$, we define the space $H(\Omega_\varepsilon)$. Mapping (4), owing to its one-to-oneness and assumed smoothness of $\boldsymbol{\Phi}_\varepsilon$, defines a one-to-one correspondence between the spaces $H(\Omega_0)$ and $H(\Omega_\varepsilon)$, i.e., if $\boldsymbol{\chi}(\boldsymbol{x}) \in H(\Omega_0)$, then $\boldsymbol{\chi}(\boldsymbol{\Phi}_\varepsilon^{-1}(\boldsymbol{y})) \in H(\Omega_\varepsilon)$, and vice versa, if $\boldsymbol{\chi}(\boldsymbol{y}) \in H(\Omega_\varepsilon)$, then $\boldsymbol{\chi}(\boldsymbol{\Phi}_\varepsilon(\boldsymbol{x})) \in H(\Omega_0)$.

Let $\boldsymbol{\nu}^\varepsilon$ be a unit vector of the normal to the perturbed crack Γ_ε. Let us define the set of admissible displacements of the body points for the perturbed problem:

$$K_\varepsilon(\Omega_\varepsilon) = \{\boldsymbol{\chi} \in H(\Omega_\varepsilon)\colon \quad [\boldsymbol{W}^{\mathrm{t}}]\boldsymbol{\nu}^\varepsilon \geq |[(\nabla w)^{\mathrm{t}}\boldsymbol{\nu}^\varepsilon]| \quad \text{almost everywhere on } \Gamma_\varepsilon\}.$$

It should be noted that the set of admissible displacements $K_0(\Omega_0)$ in the general case does not transform to the set of admissible displacements $K_\varepsilon(\Omega_\varepsilon)$, despite the one-to-one correspondence between the spaces $H(\Omega_0)$ and $H(\Omega_\varepsilon)$ owing to mapping (4). Such a transformation is not observed even for a rectilinear crack [21]. The reasons are, first, the fact that the unit normal $\boldsymbol{\nu}_0$ to Γ_0 does not transform to the unit normal $\boldsymbol{\nu}^\varepsilon$ to Γ_ε, and, second, the non-penetration condition contains a gradient operator, which changes its form under the action of the coordinate transformation (4).

Let us formulate a perturbed problem of equilibrium of a plate that occupies the perturbed domain Ω_ε as the problem of minimization of the energy functional $\Pi(\Omega_\varepsilon; \boldsymbol{\chi})$ on the set of admissible displacements $K_\varepsilon(\Omega_\varepsilon)$: we have to find a function $\boldsymbol{\chi}^\varepsilon \in K_\varepsilon(\Omega_\varepsilon)$ such that

$$\Pi(\Omega_\varepsilon; \boldsymbol{\chi}^\varepsilon) = \inf_{\boldsymbol{\chi} \in K_\varepsilon(\Omega_\varepsilon)} \Pi(\Omega_\varepsilon; \boldsymbol{\chi}), \tag{5}$$

where

$$\Pi(\Omega_\varepsilon; \boldsymbol{\chi}) = \frac{1}{2} \int\limits_{\Omega_\varepsilon} \sigma_{ij}(\boldsymbol{W})\varepsilon_{ij}(\boldsymbol{W})\, d\Omega_\varepsilon + \frac{1}{2} \int\limits_{\Omega_\varepsilon} b(w, w)\, \Omega_\varepsilon - \int\limits_{\Omega_\varepsilon} \boldsymbol{f}^{\mathrm{t}}\boldsymbol{\chi}\, d\Omega_\varepsilon.$$

Problem (5) has a unique solution, for which the following variational inequality is satisfied [3]:

$$\int\limits_{\Omega_\varepsilon} \sigma_{ij}(\boldsymbol{W}^\varepsilon)\varepsilon_{ij}(\boldsymbol{W} - \boldsymbol{W}^\varepsilon)\, d\Omega_\varepsilon + \int\limits_{\Omega_\varepsilon} b(w^\varepsilon, w - w^\varepsilon)\, d\Omega_\varepsilon \geq \int\limits_{\Omega_\varepsilon} \boldsymbol{f}^{\mathrm{t}}(\boldsymbol{\chi} - \boldsymbol{\chi}^\varepsilon)\, d\Omega \quad \forall\, \boldsymbol{\chi} \in K_\varepsilon(\Omega_\varepsilon). \tag{6}$$

The objective of the present work is to calculate the limit

$$\lim_{\varepsilon \to 0} \frac{\Pi(\Omega_\varepsilon; \boldsymbol{\chi}^\varepsilon) - \Pi(\Omega_0; \boldsymbol{\chi}_0)}{\varepsilon},$$

where $\boldsymbol{\chi}_0$ and $\boldsymbol{\chi}^\varepsilon$ are the solutions of problems (3) and (5), respectively. If this limit exists, it determined the derivative of the energy functional of the plate with respect to the perturbation parameter ε of the domain Ω_0. Its existence cannot be proved in the general case. Under certain additional restrictions on the solution $\boldsymbol{\chi}_0$ of the non-perturbed problem of equilibrium or on the perturbation velocity field (4), however, it is possible to prove the existence of the energy functional derivative.

The basic result of the present paper is the following theorem (the proof is given in Sec. 3).

Theorem 1. *If*

$$\Big\langle \sigma_\nu(\boldsymbol{W}_0), \Big|\Big[(\nabla w_0)^{\mathrm{t}}\Big(\frac{\partial \boldsymbol{V}}{\partial \boldsymbol{x}} + \Big(\frac{\partial \boldsymbol{V}}{\partial \boldsymbol{x}}\Big)^{\mathrm{t}}\Big)\nu_0\Big]\Big|\Big\rangle_{\Gamma_0} = 0, \tag{7}$$

then, for each perturbation $\boldsymbol{\Phi} \in C^1(-\varepsilon_0, \varepsilon_0; W^{2,\infty}(\mathbb{R}^N))$, *there exists the first derivative of the energy functional* $\Pi(\Omega_\varepsilon; \boldsymbol{\chi}^\varepsilon)$ *with respect to the perturbation parameter* ε *at* $\varepsilon = 0$, *which is described by the formula*

$$\frac{d\Pi(\Omega_\varepsilon; \boldsymbol{\chi}^\varepsilon)}{d\varepsilon}\Big|_{\varepsilon=0} = \int\limits_{\Omega_0} \Big(\sigma_{ij}(\boldsymbol{W}_0)\varepsilon_{ij}(\boldsymbol{Q}_0) - \bar{\boldsymbol{f}}^{\mathrm{t}}\boldsymbol{Q}_0\Big)\, d\Omega_0 + \frac{1}{2}\int\limits_{\Omega_0} A_1(\boldsymbol{V}; \boldsymbol{W}_0, \boldsymbol{W}_0)\, d\Omega_0$$

$$+ \frac{1}{2}\int\limits_{\Omega_0} A_2(\boldsymbol{V}; w_0, w_0)\, d\Omega_0 - \int\limits_{\Omega_0} \Big(\operatorname{div}(\boldsymbol{V} f_i) w_{0i} + \operatorname{div}(\boldsymbol{V} f_3) w_0\Big)\, d\Omega_0, \tag{8}$$

where $\boldsymbol{\chi}_0$ *is the solution of the non-perturbed problem* (2)*;* $\boldsymbol{Q}_0 = \boldsymbol{W}_0^{\rm t}\, \partial \boldsymbol{V}/\partial \boldsymbol{x}$*;* $A_1(\boldsymbol{V}, \boldsymbol{W}_0, \boldsymbol{W}_0)$ *and* $A_2(\boldsymbol{V}, w_0, w_0)$ *are defined below;* $\boldsymbol{V} = (V_1, V_2) = (\partial \boldsymbol{\Phi}_\varepsilon/\partial\varepsilon)\Big|_{\varepsilon=0}$*;* $\sigma_\nu(\boldsymbol{W}_0)$ *is the normal stress.*

The recording $\langle\,\cdot\,,\,\cdot\,\rangle_{\Gamma_0}$ means the duality of the spaces $H_{00}^{1/2}(\Gamma_0)$ and $(H_{00}^{1/2}(\Gamma_0))^*$. Note, if the solution is sufficiently smooth, this recording means integration over the curve Γ_0.

It follows from condition (7) that the energy functional derivative exists if the crack faces are free from stresses [in this case, $\sigma_\nu(\boldsymbol{W}_0) = 0$].

In addition, condition (7) is also valid for

$$\Big[(\nabla w_0)^{\rm t}\Big(\frac{\partial \boldsymbol{V}}{\partial \boldsymbol{x}} + \Big(\frac{\partial \boldsymbol{V}}{\partial \boldsymbol{x}}\Big)^{\rm t}\Big)\boldsymbol{\nu}_0\Big] = 0.$$

For instance, the Griffith formula for a plate with a rectilinear crack was derived in [21]: $\Gamma_0 = \{(x_1, x_2)\colon\ 0 < x_1 < l,\ x_2 = 0\}$, where $l = \text{const}$. The unit normal to such a crack is $\boldsymbol{\nu}_0 = (0, 1)$. Let consider a perturbation of the domain

$$y_1 = x_1 - \varepsilon\theta(x_1, x_2), \qquad y_2 = x_2, \tag{9}$$

where the function $\theta \in C_0^\infty(\Omega)$ is such that $\theta = 1$ in the neighborhood of the point $(l, 0)$ and $\theta = 0$ in the neighborhood of the point $(0, 0)$. Perturbation (9) corresponds to quasi-static growth of the rectilinear crack along the axis Ox_1. It can be easily noted that $\boldsymbol{V} = (-\theta, 0)$; therefore, $(\partial\boldsymbol{V}/\partial\boldsymbol{x} + (\partial\boldsymbol{V}/\partial\boldsymbol{x})^{\rm t})\boldsymbol{\nu}_0 = (-\theta_{,2}, 0)^{\rm t}$. If we impose an additional condition on θ, such that $\theta_{,2} = 0$ on Γ_0 (as in [21]), we obtain the Griffith formula for a plate with a rectilinear crack, because equality (7) is satisfied in this case.

Let us formulate and prove some auxiliary formulas and statements, which will be necessary to derive formula (8).

2. Auxiliary Statements and Formulas. We use the following notation for the functional transformation matrix (4):

$$\frac{\partial \boldsymbol{\Phi}_\varepsilon}{\partial \boldsymbol{x}} = \begin{pmatrix} \Phi^1_{\varepsilon,1} & \Phi^2_{\varepsilon,1} \\ \Phi^1_{\varepsilon,2} & \Phi^2_{\varepsilon,2} \end{pmatrix}.$$

We determine the vectors corresponding to the first-order and second-order partial derivatives:

$$\frac{\partial}{\partial \boldsymbol{x}} = \Big(\frac{\partial}{\partial x_1}\ \frac{\partial}{\partial x_2}\Big)^{\rm t}, \qquad \frac{\partial^2}{\partial \boldsymbol{x}^2} = \Big(\frac{\partial^2}{\partial x_1^2}\ \frac{\partial^2}{\partial x_1\,\partial x_2}\ \frac{\partial^2}{\partial x_2^2}\Big)^{\rm t}.$$

The vectors $\partial/\partial\boldsymbol{y}$ and $\partial^2/\partial\boldsymbol{y}^2$ are defined in a similar manner. Let us define the functional matrices A and $\partial^2\Phi_\varepsilon/\partial x^2$:

$$A = \begin{pmatrix} \Phi^1_{\varepsilon,11} & \Phi^2_{\varepsilon,11} \\ \Phi^1_{\varepsilon,12} & \Phi^2_{\varepsilon,12} \\ \Phi^1_{\varepsilon,22} & \Phi^2_{\varepsilon,22} \end{pmatrix}, \qquad \frac{\partial^2\Phi_\varepsilon}{\partial x^2} = \begin{pmatrix} (\Phi^1_{\varepsilon,1})^2 & 2\Phi^1_{\varepsilon,1}\Phi^2_{\varepsilon,1} & (\Phi^2_{\varepsilon,1})^2 \\ \Phi^1_{\varepsilon,1}\Phi^1_{\varepsilon,2} & \Phi^1_{\varepsilon,1}\Phi^2_{\varepsilon,2} + \Phi^1_{\varepsilon,2}\Phi^2_{\varepsilon,1} & \Phi^2_{\varepsilon,1}\Phi^2_{\varepsilon,2} \\ (\Phi^1_{\varepsilon,2})^2 & 2\Phi^1_{\varepsilon,2}\Phi^2_{\varepsilon,2} & (\Phi^2_{\varepsilon,2})^2 \end{pmatrix}.$$

Using the above-introduced notations, we can write the transformation of the derivatives in a matrix form as

$$\frac{\partial}{\partial \boldsymbol{x}} = \frac{\partial \boldsymbol{\Phi}_\varepsilon}{\partial \boldsymbol{x}}\,\frac{\partial}{\partial \boldsymbol{y}}, \qquad \frac{\partial^2}{\partial \boldsymbol{x}^2} = \frac{\partial^2\Phi_\varepsilon}{\partial x^2}\,\frac{\partial^2}{\partial \boldsymbol{y}^2} + A\,\frac{\partial}{\partial \boldsymbol{y}}. \tag{10}$$

By virtue of smoothness of $\boldsymbol{\Phi}_\varepsilon$, the following expansion with respect to ε is valid:

$$\boldsymbol{\Phi}_\varepsilon(\boldsymbol{x}) = \boldsymbol{x} + \varepsilon\boldsymbol{V}(\boldsymbol{x}) + \boldsymbol{r}_1(\varepsilon, \boldsymbol{x}) \qquad \text{in } \mathbb{R}^2. \tag{11}$$

Here the function $\boldsymbol{r}_1(\varepsilon, \boldsymbol{x}) \in C(-\varepsilon_0, \varepsilon_0; [W^{2,\infty}(\mathbb{R}^2)]^2)$ and $\boldsymbol{r}_1(\varepsilon, \boldsymbol{x})/\varepsilon \to 0$ strongly in $[W^{2,\infty}_{\rm loc}(\mathbb{R}^2)]^2$ as $\varepsilon \to 0$.

It follows from Eq. (11) that the Jacobian $J_\varepsilon(\boldsymbol{x})$ of transformation (4) admits the presentation

$$J_\varepsilon(\boldsymbol{x}) \equiv \Big|\frac{\partial \boldsymbol{\Phi}_\varepsilon}{\partial \boldsymbol{x}}\Big| = 1 + \varepsilon\,\mathrm{div}\,\boldsymbol{V} + r_2(\varepsilon, \boldsymbol{x}) \qquad \text{in } \mathbb{R}^2,$$

where $r_2 \in C(-\varepsilon_0, \varepsilon_0; W^{1,\infty}(\mathbb{R}^N))$ and $r_2(\varepsilon, \boldsymbol{x})/\varepsilon \to 0$ strongly in $W^{1,\infty}_{\rm loc}(\mathbb{R}^2)$ as $\varepsilon \to 0$. It should be noted that the Jacobian $J_\varepsilon(\boldsymbol{x})$ is rigorously positive for small values of ε.

In turn, the functional matrices $\partial\boldsymbol{\Phi}_\varepsilon/\partial\boldsymbol{x}$, A, and $\partial^2\Phi_\varepsilon/\partial x^2$ admit the presentation

$$\frac{\partial \boldsymbol{\Phi}_\varepsilon}{\partial \boldsymbol{x}} = I + \varepsilon\,\frac{\partial \boldsymbol{V}}{\partial \boldsymbol{x}} + r_3(\varepsilon, \boldsymbol{x}), \qquad \|r_3(\varepsilon, \boldsymbol{x})\|_{[W^{1,\infty}_{\rm loc}(\mathbb{R}^2)]^4} = o(\varepsilon),$$

$$A = \varepsilon \begin{pmatrix} V_{1,11} & V_{2,11} \\ V_{1,12} & V_{2,12} \\ V_{1,22} & V_{2,22} \end{pmatrix} + r_4(\varepsilon, \boldsymbol{x}), \qquad \|r_4(\varepsilon, \boldsymbol{x})\|_{[L^\infty_{\text{loc}}(\mathbb{R}^2)]^6} = o(\varepsilon), \tag{12}$$

$$\frac{\partial^2 \Phi_\varepsilon}{\partial x^2} = I + \varepsilon \begin{pmatrix} 2V_{1,1} & 2V_{2,1} & 0 \\ V_{1,2} & V_{1,1} + V_{2,2} & V_{2,1} \\ 0 & 2V_{1,2} & 2V_{2,2} \end{pmatrix} + r_5(\varepsilon, \boldsymbol{x}), \qquad \|r_5(\varepsilon, \boldsymbol{x})\|_{[W^{1,\infty}_{\text{loc}}(\mathbb{R}^2)]^9} = o(\varepsilon)$$

almost everywhere in Ω_0 (I is the unit matrix). The determinant of the matrix $\partial^2\Phi_\varepsilon/\partial x^2$ can be expanded with respect to ε as

$$j_\varepsilon(\boldsymbol{x}) = \left|\frac{\partial^2 \Phi_\varepsilon}{\partial x^2}\right| = 1 + 3\varepsilon \operatorname{div} \boldsymbol{V}(\boldsymbol{x}) + r_6(\varepsilon, \boldsymbol{x}), \qquad \|r_6(\varepsilon, \boldsymbol{x})\|_{L^\infty_{\text{loc}}(\mathbb{R}^2)} = o(\varepsilon) \qquad \text{almost everywhere in } \Omega_0.$$

For sufficiently small values of ε, the determinant $j_\varepsilon(\boldsymbol{x})$ is rigorously positive; hence, there exists an inverse functional matrix $\psi = (\partial^2\Phi_\varepsilon/\partial x^2)^{-1}$. By virtue of Eq. (10) and $J_\varepsilon(\boldsymbol{x}) > 0$ and $j_\varepsilon(\boldsymbol{x}) > 0$, the inverse transformation of derivatives can be written in a matrix form as

$$\frac{\partial}{\partial \boldsymbol{y}} = \Psi \frac{\partial}{\partial \boldsymbol{x}}, \qquad \frac{\partial^2}{\partial \boldsymbol{y}^2} = \psi \frac{\partial^2}{\partial \boldsymbol{x}^2} + a \frac{\partial}{\partial \boldsymbol{x}}, \tag{13}$$

where $\Psi = (\partial \boldsymbol{\Phi}_\varepsilon/\partial \boldsymbol{x})^{-1}$ and $a = -(\partial^2\Phi_\varepsilon/\partial x^2)^{-1} A (\partial \boldsymbol{\Phi}_\varepsilon/\partial \boldsymbol{x})^{-1}$. By virtue of Eqs. (12), the matrices Ψ, a, and ψ can be expanded with respect to ε as

$$\Psi = I - \varepsilon \frac{\partial \boldsymbol{V}}{\partial \boldsymbol{x}} + r_7(\varepsilon, \boldsymbol{x}), \qquad \|r_7(\varepsilon, \boldsymbol{x})\|_{[W^{1,\infty}_{\text{loc}}(\mathbb{R}^2)]} = o(\varepsilon),$$

$$a = -\varepsilon \begin{pmatrix} V_{1,11} & V_{2,11} \\ V_{1,12} & V_{2,12} \\ V_{1,22} & V_{2,22} \end{pmatrix} + r_8(\varepsilon, \boldsymbol{x}), \qquad \|r_8(\varepsilon, \boldsymbol{x})\|_{[L^\infty_{\text{loc}}(\mathbb{R}^2)]^6} = o(\varepsilon), \tag{14}$$

$$\psi = I - \varepsilon \begin{pmatrix} 2V_{1,1} & 2V_{2,1} & 0 \\ V_{1,2} & V_{1,1} + V_{2,2} & V_{2,1} \\ 0 & 2V_{1,2} & 2V_{2,2} \end{pmatrix} + r_9(\varepsilon, \boldsymbol{x}), \qquad \|r_9(\varepsilon, \boldsymbol{x})\|_{[W^{1,\infty}_{\text{loc}}(\mathbb{R}^2)]^9} = o(\varepsilon)$$

almost everywhere in Ω_0.

We denote the matrices for $-\varepsilon$ in Eqs. (14) as $\bar{a}(\boldsymbol{V})$ and $\bar{\psi}(\boldsymbol{V})$, respectively. Let us determine the nondegenerate constant matrix K:

$$K = \begin{pmatrix} 1 & 0 & k \\ 0 & 2(1-k) & 0 \\ k & 0 & 1 \end{pmatrix}.$$

Then, the bilinear form $b(u, v)$ can be recorded in a matrix form

$$b(u, v) = \Big(\frac{\partial^2 u}{\partial \boldsymbol{x}^2}\Big)^{\mathrm{t}} K \frac{\partial^2 v}{\partial \boldsymbol{x}^2}.$$

Applying the inverse transformation to the domain Ω_ε, we can show that the crack Γ_ε transforms to the crack Γ_0. The normal vector $\boldsymbol{\nu}^\varepsilon$ transforms to a new vector $\boldsymbol{\nu}_\varepsilon$ determined on Γ_0, which does not coincide in the general case with the normal vector $\boldsymbol{\nu}_0$ to Γ_0. The set $K_\varepsilon(\Omega_\varepsilon)$ experiences a one-to-one transformation to a new set

$$K_\varepsilon(\Omega_0) = \{\boldsymbol{\chi} \in H(\Omega_0)\colon \quad [\boldsymbol{W}^{\mathrm{t}}]\boldsymbol{\nu}_\varepsilon \geq |[(\nabla w)^{\mathrm{t}} \Psi^{\mathrm{t}} \boldsymbol{\nu}_\varepsilon]| \quad \text{almost everywhere on } \Gamma_0\}.$$

After that, applying the coordinate transformation (4) to functions and integrals involved in the variational inequality (6), we obtain the variational inequality

$$\int\limits_{\Omega_0} J_\varepsilon c_{ijkl} E_{kl}(\Psi; \boldsymbol{W}_\varepsilon) E_{ij}(\Psi; \boldsymbol{W} - \boldsymbol{W}_\varepsilon)\, d\Omega_0$$

$$+ \int\limits_{\Omega_0} J_\varepsilon \Big(\psi \frac{\partial^2 w_\varepsilon}{\partial \boldsymbol{x}^2} + a \frac{\partial w_\varepsilon}{\partial \boldsymbol{x}}\Big)^{\mathrm{t}} K \Big(\psi \frac{\partial^2 (w - w_\varepsilon)}{\partial \boldsymbol{x}^2} + a \frac{\partial\, (w - w_\varepsilon)}{\partial \boldsymbol{x}}\Big)\, d\Omega_0 \int\limits_{\Omega_0} J_\varepsilon \boldsymbol{f}^{\varepsilon\,\mathrm{t}} (\boldsymbol{\chi} - \boldsymbol{\chi}_\varepsilon)\, d\Omega_0 \qquad \forall \boldsymbol{\chi} \in K_\varepsilon(\Omega_0), \tag{15}$$

where $\boldsymbol{\chi}_\varepsilon(\boldsymbol{x}) = \boldsymbol{\chi}^\varepsilon(\boldsymbol{\Phi}_\varepsilon(\boldsymbol{x}))$, i.e., the function $\boldsymbol{\chi}_\varepsilon(\boldsymbol{x})$, which belongs to $K_\varepsilon(\Omega_0)$, is the solution $\boldsymbol{\chi}^\varepsilon(\boldsymbol{y}) \in K_\varepsilon(\Omega_\varepsilon)$ of the perturbed problem (5) written in non-perturbed coordinates. Similarly, for $\boldsymbol{f}$, we obtain $\boldsymbol{f}^\varepsilon(\boldsymbol{x}) = \boldsymbol{f}(\boldsymbol{\Phi}_\varepsilon(\boldsymbol{x}))$, $\boldsymbol{x} \in \Omega_0$. In Eq. (15), $E_{ij}(\Psi; \boldsymbol{W})$ is the transformed strain tensor

$$E_{ij}(\Psi; \boldsymbol{W}) = \frac{1}{2}\Big(\frac{\partial u_i}{\partial x_k}\,\Psi_{kj} + \frac{\partial u_j}{\partial x_k}\,\Psi_{ki}\Big),$$

$\{c_{ijkl}\}$ is the tensor of elasticity coefficients, such that $c_{1111} = c_{2222} = 1$, $c_{1122} = c_{2211} = k$, and $c_{1212} = c_{1221} = c_{2112} = c_{2121} = (1-k)/2$, and the remaining coefficients are equal to zero.

Thus, the following theorem is valid.

Theorem 2. *For sufficiently small values of ε, the solution $\boldsymbol{\chi}^\varepsilon \in K_\varepsilon(\Omega_\varepsilon)$ of the perturbed problem* (6) *mapped onto the initial domain Ω_0 with the help of the inverse transformation to* (4) *is a unique solution $\boldsymbol{\chi}_\varepsilon \in K_\varepsilon(\Omega_0)$ of the variational inequality* (15).

Let us substitute expansion (14) into inequality (15). As a result, we obtain

$$\int\limits_{\Omega_0} J_\varepsilon c_{ijkl} E_{kl}(\Psi; \boldsymbol{U}) E_{ij}(\Psi; \boldsymbol{W})\, d\Omega_0$$

$$= \int\limits_{\Omega_0} \Big(\sigma_{ij}(\boldsymbol{U})\varepsilon_{ij}(\boldsymbol{W}) + \varepsilon A_1(\boldsymbol{V}; \boldsymbol{U}, \boldsymbol{W})\Big)\, d\Omega_0 + o(\varepsilon) R_1(\boldsymbol{U}, \boldsymbol{W}); \tag{16}$$

$$\int\limits_{\Omega_0} J_\varepsilon \Big(\psi\,\frac{\partial^2 u}{\partial \boldsymbol{x}^2} + a\,\frac{\partial u}{\partial \boldsymbol{x}}\Big)^{\mathrm{t}} K \Big(\psi\,\frac{\partial^2 v}{\partial \boldsymbol{x}^2} + a\,\frac{\partial v}{\partial \boldsymbol{x}}\Big)\, d\Omega_0$$

$$= \int\limits_{\Omega_0} \Big(b(u, v) + \varepsilon A_2(\boldsymbol{V}; u, v)\Big)\, d\Omega_0 + o(\varepsilon) R_2(u, v); \tag{17}$$

$$\int\limits_{\Omega_0} J_\varepsilon (\boldsymbol{f}^\varepsilon)^{\mathrm{t}} \boldsymbol{\chi}\, d\Omega_0 = \int\limits_{\Omega_0} \Big(f^{\mathrm{t}}\boldsymbol{\chi} + \varepsilon\, \mathrm{div}\,(\boldsymbol{V} f_i) w_i + \varepsilon\, \mathrm{div}\,(\boldsymbol{V} f_3) w\Big)\, d\Omega_0 + o(\varepsilon) R_3(\boldsymbol{\chi}), \tag{18}$$

where

$$A_1(\boldsymbol{V}; \boldsymbol{U}, \boldsymbol{W}) = \mathrm{div}\,\boldsymbol{V} \sigma_{ij}(\boldsymbol{U})\varepsilon_{ij}(\boldsymbol{W}) - \sigma_{ij}(\boldsymbol{U}) E_{ij}\Big(\frac{\partial \boldsymbol{V}}{\partial \boldsymbol{x}}; \boldsymbol{W}\Big) - \sigma_{ij}(\boldsymbol{W}) E_{ij}\Big(\frac{\partial \boldsymbol{V}}{\partial \boldsymbol{x}}; \boldsymbol{U}\Big),$$

$$A_2(\boldsymbol{V}; u, v) = b(u, v)\,\mathrm{div}\,\boldsymbol{V} - \Big(\frac{\partial^2 u}{\partial \boldsymbol{x}^2}\Big)^{\mathrm{t}} \Big(K\bar{\psi}(\boldsymbol{V}) + \bar{\psi}^{\mathrm{t}}(\boldsymbol{V})K\Big)\,\frac{\partial^2 v}{\partial \boldsymbol{x}^2} \tag{19}$$

$$- \Big(\frac{\partial^2 u}{\partial \boldsymbol{x}^2}\Big)^{\mathrm{t}} K\bar{a}(\boldsymbol{V})\,\frac{\partial v}{\partial \boldsymbol{x}} - \Big(\frac{\partial u}{\partial \boldsymbol{x}}\Big)^{\mathrm{t}} \bar{a}^{\mathrm{t}}(\boldsymbol{V})K\,\frac{\partial^2 v}{\partial \boldsymbol{x}^2},$$

R_1, R_2, and R_3 are certain bounded polylinear forms.

Let us substitute the test functions $\boldsymbol{\chi} = 0$ and $\boldsymbol{\chi} = 2\boldsymbol{\chi}_\varepsilon$ into inequality (15). Then, using Eqs. (16)–(19), we obtain the following estimate uniform with respect to ε:

$$\|\boldsymbol{\chi}_\varepsilon\|_{H(\Omega_0)} \le c.$$

Let us consider the crack Γ_0. After transformation (4) is applied, the curve Γ_0 transforms to the curve Γ_ε described by the equation

$$\overline{\Upsilon}(\boldsymbol{y}) = \Upsilon(\boldsymbol{\Phi}_\varepsilon^{-1}(\boldsymbol{y})) = 0, \qquad \boldsymbol{y} \in \Gamma_\varepsilon.$$

The unit normal $\boldsymbol{\nu}^\varepsilon$ to Γ_ε can be found from the formula

$$\boldsymbol{\nu}^\varepsilon = \nabla_y \overline{\Upsilon} / |\nabla_y \overline{\Upsilon}|.$$

After the use of Eq. (13), the transformed vector of the unit normal $\boldsymbol{\nu}_\varepsilon$ determined on Γ_0 acquires the form

$$\boldsymbol{\nu}_\varepsilon = \Psi \nabla_x \Upsilon / |\Psi \nabla_x \Upsilon|.$$

As mapping (4) is nondegenerate and the curve Γ_0 is regular, then $|\nabla_x \Upsilon| \neq 0$ and $|\Psi\nabla_x\Upsilon| \neq 0$. Moreover, $\nabla_x\Upsilon = \boldsymbol{\nu}_0|\nabla_x\Upsilon|$; hence, the set $K_\varepsilon(\Omega_0)$ can be determined in an equivalent form as

$$K_\varepsilon(\Omega_0) = \{\boldsymbol{\chi} \in H(\Omega_0)\colon \quad [\boldsymbol{W}^{\mathrm{t}}]\Psi\boldsymbol{\nu}_0 \geq |[(\nabla w)^{\mathrm{t}}\Psi^{\mathrm{t}}\Psi\boldsymbol{\nu}_0]| \quad \text{almost everywhere on } \Gamma_0\}.$$

Let us consider an arbitrary function $\boldsymbol{\chi} = (\boldsymbol{W}, w)$ belonging to the set $K_\varepsilon(\Omega_0)$. This function satisfies the condition

$$[\boldsymbol{W}^{\mathrm{t}}]\Psi\boldsymbol{\nu}_0 \geq |[(\nabla w)^{\mathrm{t}}\Psi^{\mathrm{t}}\Psi\boldsymbol{\nu}_0]| \qquad \text{almost everywhere on } \Gamma_0. \tag{20}$$

Substituting expansion (14) of the matrix Ψ into Eq. (20), we obtain

$$[\boldsymbol{W}^{\mathrm{t}}]\boldsymbol{\nu}_0 - \varepsilon[\boldsymbol{W}^{\mathrm{t}}]\Big(\frac{\partial \boldsymbol{V}}{\partial \boldsymbol{x}} - \frac{r_7(\varepsilon)}{\varepsilon}\Big)\boldsymbol{\nu}_0$$

$$\geq \Big|\Big[(\nabla w)^{\mathrm{t}}\boldsymbol{\nu}_0 - \varepsilon(\nabla w)^{\mathrm{t}}\Big(\frac{\partial \boldsymbol{V}}{\partial \boldsymbol{x}} + \Big(\frac{\partial \boldsymbol{V}}{\partial \boldsymbol{x}}\Big)^{\mathrm{t}} + \frac{r_{10}(\varepsilon)}{\varepsilon}\Big)\boldsymbol{\nu}_0\Big]\Big| \quad \text{almost everywhere on } \Gamma_0, \tag{21}$$

where

$$r_{10}(\varepsilon) = \varepsilon^2\Big(\frac{\partial \boldsymbol{V}}{\partial \boldsymbol{x}}\Big)^{\mathrm{t}}\frac{\partial \boldsymbol{V}}{\partial \boldsymbol{x}} - \varepsilon(r_7(\varepsilon))^{\mathrm{t}}\frac{\partial \boldsymbol{V}}{\partial \boldsymbol{x}} - \varepsilon\Big(\frac{\partial \boldsymbol{V}}{\partial \boldsymbol{x}}\Big)^{\mathrm{t}} r_7(\varepsilon) + r_7(\varepsilon) + (r_7(\varepsilon))^{\mathrm{t}} + (r_7(\varepsilon))^{\mathrm{t}} r_7(\varepsilon).$$

In this case, we have $\|r_{10}(\varepsilon)\|_{[W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^2)]^4} = o(\varepsilon)$.

Let us prove some auxiliary lemmas.

Lemma 1. *Let $\boldsymbol{\chi}_0 \in K_0(\Omega_0)$ be a solution of the non-perturbed problem* (3) *and $\boldsymbol{\chi}_\varepsilon \in K_\varepsilon(\Omega_0)$ be a solution of problem* (15)*. Then, there exist functions $\boldsymbol{\lambda}^1_\varepsilon$ and $\boldsymbol{\lambda}^2_\varepsilon$ such that*

$$\|\boldsymbol{\lambda}^i_\varepsilon\|_{H(\Omega_0)} \leq c \qquad (i = 1, 2)$$

is uniform with respect to ε, and the following inclusions are valid:

$$\boldsymbol{\chi}^1_\varepsilon = \boldsymbol{\chi}_0 + \varepsilon\boldsymbol{\lambda}^1_\varepsilon \in K_\varepsilon(\Omega_0), \qquad \boldsymbol{\chi}^2_\varepsilon = \boldsymbol{\chi}_\varepsilon + \varepsilon\boldsymbol{\lambda}^2_\varepsilon \in K_0(\Omega_0).$$

Proof. Let us consider a simply connected domain O with a smooth boundary γ, such that $\overline{O} \subset \Omega$, Γ_0 is a part of γ, and the outward normal to γ coincides with $\boldsymbol{\nu}_0$ on Γ_0. Let us introduce the following notation:

$$g = -\Big|\Big[(\nabla w_0)^{\mathrm{t}}\Big(\frac{\partial \boldsymbol{V}}{\partial \boldsymbol{x}} + \Big(\frac{\partial \boldsymbol{V}}{\partial \boldsymbol{x}}\Big)^{\mathrm{t}} + \frac{r_{10}(\varepsilon)}{\varepsilon}\Big)\boldsymbol{n}\Big]\Big|.$$

The function $w_0 \in H^{2,0}(\Omega_0)$; hence, $[\nabla w_0] \in H^{1/2}(\gamma)$. In addition, $\partial\boldsymbol{V}/\partial\boldsymbol{x}$ and $r_{10}(\varepsilon)$ belong to $[W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^2)]^4 \subset [C^{0,1}_{\mathrm{loc}}(\mathbb{R}^2)]^4$ [30], $\boldsymbol{n} \in C^{0,1}(\gamma)$. Then, $g \in H^{1/2}(\gamma)$, and $g \equiv 0$ outside Γ_0 [3]. As the components of the normal $\boldsymbol{n}$ belong to $C^{0,1}(\gamma)$, then $g\boldsymbol{n} \in [H^{1/2}(\gamma)]^2$. Hence, there exists a function $\boldsymbol{W}^1_\varepsilon \in [H^1(O)]^2$ such that its trace on γ equals $g\boldsymbol{n}$. Let $\boldsymbol{W}^1_\varepsilon \equiv 0$ outside O. Let θ be an infinitely differentiable finite function in Ω, such that $\theta = 1$ on Γ_0.

Let us consider the function $\tilde{\boldsymbol{W}}^1_\varepsilon = \theta\boldsymbol{W}^1_\varepsilon$. By construction, $\tilde{\boldsymbol{W}}^1_\varepsilon \in H^{1,0}(\Omega_0) \times H^{1,0}(\Omega_0)$ and

$$[(\tilde{\boldsymbol{W}}^1_\varepsilon)^{\mathrm{t}}]\boldsymbol{\nu}_0 = \Big|\Big[(\nabla w_0)^{\mathrm{t}}\Big(\frac{\partial \boldsymbol{V}}{\partial \boldsymbol{x}} + \Big(\frac{\partial \boldsymbol{V}}{\partial \boldsymbol{x}}\Big)^{\mathrm{t}} + \frac{r_{10}(\varepsilon)}{\varepsilon}\Big)\boldsymbol{\nu}_0\Big]\Big| \qquad \text{almost everywhere on } \Gamma_0.$$

Then, we consider the matrix equation

$$\boldsymbol{U}^{\mathrm{t}} A = \boldsymbol{b}^{\mathrm{t}}, \tag{22}$$

where

$$A = I - \varepsilon\Big(\frac{\partial \boldsymbol{V}}{\partial \boldsymbol{x}} - \frac{r_7(\varepsilon)}{\varepsilon}\Big), \qquad \boldsymbol{b}^{\mathrm{t}} = \boldsymbol{W}_0^{\mathrm{t}}\Big(\frac{\partial \boldsymbol{V}}{\partial \boldsymbol{x}} - \frac{r_7(\varepsilon)}{\varepsilon}\Big) + (\tilde{\boldsymbol{W}}^1_\varepsilon)^{\mathrm{t}}, \tag{23}$$

$\boldsymbol{\chi}_0 = (\boldsymbol{W}_0, w_0)$ is the solution of the non-perturbed problem. The elements of the matrix A belong to the space $W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^2)$, and the vector is $\boldsymbol{b} \in H^{1,0}(\Omega_0) \times H^{1,0}(\Omega_0)$. In addition, for all sufficiently small values of ε, the determinant $|A|$ of the matrix A in Ω_0 differs from zero. Hence, Eq. (22) has a unique solution almost for all $x \in \Omega_0$.

Let $\boldsymbol{U}^1_\varepsilon$ be a solution of this equation. We will show that $(\boldsymbol{U}^1_\varepsilon, 0) \in H(\Omega_0)$. Indeed, the elements of the matrix A belong to $W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^2)$; hence, they belong to the space $C^{0,1}_{\mathrm{loc}}(\mathbb{R}^2)$ [30] and, consequently, $|A| \in C^{0,1}_{\mathrm{loc}}(\mathbb{R}^2)$. As $|A|(\boldsymbol{x})$ does not vanish for all $\boldsymbol{x} \in \Omega_0$, then $1/|A| \in C^{0,1}_{\mathrm{loc}}(\mathbb{R}^2)$, and hence, $1/|A| \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^2)$ [30]. Thus, the elements of the inverse matrix A^{-1} belong to the space $W^{1,\infty}_{\text{loc}}(\mathbb{R}^2)$. The elements of the vector-function $\boldsymbol{b}$ belong to the space $H^{1,0}(\Omega_0)$; therefore, the solution $\boldsymbol{U}^1_\varepsilon$ of system (22) belongs to $H^{1,0}(\Omega_0) \times H^{1,0}(\Omega_0)$.

Let us assume that $\boldsymbol{\lambda}^1_\varepsilon = (\boldsymbol{U}^1_\varepsilon, 0)$ and show that $\boldsymbol{\chi}^1_\varepsilon = \boldsymbol{\chi}_0 + \varepsilon\boldsymbol{\lambda}^1_\varepsilon \in K_\varepsilon(\Omega_0)$. For this purpose, it is sufficient to verify that $\boldsymbol{\chi}^1_\varepsilon$ satisfies condition (21).

We substitute the function $\boldsymbol{\chi}^1_\varepsilon$ into Eq. (21). Taking into account that $\boldsymbol{U}^1_\varepsilon$ is a solution of Eq. (22), we obtain the following chain of equalities and inequalities:

$$[\boldsymbol{W}^{\rm t}_0]\boldsymbol{\nu}_0 + \varepsilon[(\boldsymbol{U}^1_\varepsilon)^{\rm t}]\boldsymbol{\nu}_0 - \varepsilon[\boldsymbol{W}^{\rm t}_0]\Big(\frac{\partial \boldsymbol{V}}{\partial \boldsymbol{x}} - \frac{r_7(\varepsilon)}{\varepsilon}\Big)\boldsymbol{\nu}_0 - \varepsilon^2[(\boldsymbol{U}^1_\varepsilon)^{\rm t}]\Big(\frac{\partial \boldsymbol{V}}{\partial \boldsymbol{x}} - \frac{r_7(\varepsilon)}{\varepsilon}\Big)\boldsymbol{\nu}_0$$

$$= [\boldsymbol{W}^{\rm t}_0]\boldsymbol{\nu}_0 + \varepsilon[(\tilde{\boldsymbol{W}}^1_\varepsilon)^{\rm t}]\boldsymbol{\nu}_0 = [\boldsymbol{W}^{\rm t}_0]\boldsymbol{\nu}_0 + \varepsilon\Big|\Big[(\nabla w_0)^{\rm t}\Big(\frac{\partial \boldsymbol{V}}{\partial \boldsymbol{x}} + \Big(\frac{\partial \boldsymbol{V}}{\partial \boldsymbol{x}}\Big)^{\rm t} + \frac{r_{10}(\varepsilon)}{\varepsilon}\Big)\boldsymbol{\nu}_0\Big]\Big|$$

$$\geq \Big|\Big[(\nabla w_0)^{\rm t}\boldsymbol{\nu}_0 - \varepsilon(\nabla w_0)^{\rm t}\Big(\frac{\partial \boldsymbol{V}}{\partial \boldsymbol{x}} + \Big(\frac{\partial \boldsymbol{V}}{\partial \boldsymbol{x}}\Big)^{\rm t} + \frac{r_{10}(\varepsilon)}{\varepsilon}\Big)\boldsymbol{\nu}_0\Big]\Big| \qquad \text{almost everywhere on } \Gamma_0.$$

This means that $\boldsymbol{\chi}^1_\varepsilon = \boldsymbol{\chi}_0 + \varepsilon(\boldsymbol{U}^1_\varepsilon, 0) \in K_\varepsilon(\Omega_0)$ for all sufficiently small values of ε.

Let us construct a sequence $\boldsymbol{\lambda}^2_\varepsilon$. Similar to the function $\tilde{\boldsymbol{W}}^1_\varepsilon$, we can construct a function $\tilde{\boldsymbol{W}}^2_\varepsilon \in H^{1,0}(\Omega_0) \times H^{1,0}(\Omega_0)$ such that

$$[(\tilde{\boldsymbol{W}}^2_\varepsilon)^{\rm t}]\boldsymbol{\nu}_0 = \Big|\Big[(\nabla w_\varepsilon)^{\rm t}\Big(\frac{\partial \boldsymbol{V}}{\partial \boldsymbol{x}} + \Big(\frac{\partial \boldsymbol{V}}{\partial \boldsymbol{x}}\Big)^{\rm t} + \frac{r_{10}(\varepsilon)}{\varepsilon}\Big)\boldsymbol{\nu}_0\Big]\Big| \qquad \text{almost everywhere on } \Gamma_0.$$

We assume that $(\boldsymbol{U}^2_\varepsilon)^{\rm t} = -\boldsymbol{W}^{\rm t}_\varepsilon(\partial\boldsymbol{V}/\partial\boldsymbol{x} - r_7(\varepsilon)/\varepsilon) + (\tilde{\boldsymbol{W}}^2_\varepsilon)^{\rm t}$ and $\boldsymbol{\lambda}^2_\varepsilon = (\boldsymbol{U}^2_\varepsilon, 0)$. Let us show that the function is $\boldsymbol{\chi}^2_\varepsilon = \boldsymbol{\chi}_\varepsilon + \varepsilon\boldsymbol{\lambda}^2_\varepsilon \in K_0(\Omega_0)$. Obviously, $\boldsymbol{\chi}^2_\varepsilon \in H(\Omega_0)$; therefore, it is sufficient to check the condition on Γ_0:

$$[\boldsymbol{W}^{\rm t}]\boldsymbol{\nu}_0 \geq |[(\nabla w)^{\rm t}\boldsymbol{\nu}_0]| \qquad \text{almost everywhere on } \Gamma_0.$$

As a result, we obtain

$$[\boldsymbol{W}^{\rm t}_\varepsilon]\boldsymbol{\nu}_0 - \varepsilon[\boldsymbol{W}^{\rm t}_\varepsilon]\Big(\frac{\partial \boldsymbol{V}}{\partial \boldsymbol{x}} - \frac{r_7(\varepsilon)}{\varepsilon}\Big)\boldsymbol{\nu}_0 + [(\tilde{\boldsymbol{W}}^2_\varepsilon)^{\rm t}]\boldsymbol{\nu}_0$$

$$= [\boldsymbol{W}^{\rm t}_\varepsilon]\boldsymbol{\nu}_0 - \varepsilon[\boldsymbol{W}^{\rm t}_\varepsilon]\Big(\frac{\partial \boldsymbol{V}}{\partial \boldsymbol{x}} - \frac{r_7(\varepsilon)}{\varepsilon}\Big)\boldsymbol{\nu}_0 + \Big|\Big[(\nabla w_\varepsilon)^{\rm t}\Big(\frac{\partial \boldsymbol{V}}{\partial \boldsymbol{x}} + \Big(\frac{\partial \boldsymbol{V}}{\partial \boldsymbol{x}}\Big)^{\rm t} + \frac{r_{10}(\varepsilon)}{\varepsilon}\Big)\boldsymbol{\nu}_0\Big]\Big|$$

$$\geq \Big|\Big[(\nabla w_\varepsilon)^{\rm t}\boldsymbol{\nu}_0 - \varepsilon(\nabla w_\varepsilon)^{\rm t}\Big(\frac{\partial \boldsymbol{V}}{\partial \boldsymbol{x}} + \Big(\frac{\partial \boldsymbol{V}}{\partial \boldsymbol{x}}\Big)^{\rm t} + \frac{r_{10}(\varepsilon)}{\varepsilon}\Big)\boldsymbol{\nu}_0\Big]\Big|$$

$$+ \Big|\Big[(\nabla w_\varepsilon)^{\rm t}\Big(\frac{\partial \boldsymbol{V}}{\partial \boldsymbol{x}} + \Big(\frac{\partial \boldsymbol{V}}{\partial \boldsymbol{x}}\Big)^{\rm t} + \frac{r_{10}(\varepsilon)}{\varepsilon}\Big)\boldsymbol{\nu}_0\Big]\Big| \geq |[(\nabla w_\varepsilon)^{\rm t}\boldsymbol{\nu}_0]| \qquad \text{almost everywhere on } \Gamma_0.$$

The last inequality implies that $\boldsymbol{\chi}^2_\varepsilon = \boldsymbol{\chi}_\varepsilon + \varepsilon\boldsymbol{\lambda}^2_\varepsilon \in K_0(\Omega_0)$. Lemma 1 is proved.

Theorem 3. *Let $\boldsymbol{\chi}_\varepsilon$ be a solution of problem* (15), *and $\boldsymbol{\chi}_0$ be a solution of problem* (3). *Then, we have*

$$\|\boldsymbol{\chi}_\varepsilon - \boldsymbol{\chi}_0\|_{H(\Omega_0)} \leq c\varepsilon, \tag{24}$$

where the constant c is independent of ε.

Proof. By virtue of Lemma 1, the functions $\boldsymbol{\chi}^2_\varepsilon \in K_0(\Omega_0)$ and $\boldsymbol{\chi}^1_\varepsilon \in K_\varepsilon(\Omega_0)$ can be substituted into the variational inequalities (3) and (15), respectively, as test functions. Summing up the resultant inequalities and using expansions (16)–(18) and Korn's inequalities [31], we obtain estimate (24). Theorem 1 is proved.

Let us consider the functions $\tilde{\boldsymbol{W}}^1_\varepsilon$ and $\tilde{\boldsymbol{W}}^2_\varepsilon$ constructed within Lemma 1. By virtue of the strong convergence of the sequence $\boldsymbol{\chi}_\varepsilon$ to the solution of the non-perturbed problem $\boldsymbol{\chi}_0$ and continuity of imbedding of the space $H^1(O)$ in $H^{1/2}(\gamma)$, the traces of the functions $\tilde{\boldsymbol{W}}^1_\varepsilon$ and $\tilde{\boldsymbol{W}}^2_\varepsilon$ strongly converge to $g_0\boldsymbol{\nu}_0$ in $H^{1/2}(\Gamma^-_0)$, where

$$g_0 = -\Big|\Big[(\nabla w_0)^{\rm t}\Big(\frac{\partial \boldsymbol{V}}{\partial \boldsymbol{x}} + \Big(\frac{\partial \boldsymbol{V}}{\partial \boldsymbol{x}}\Big)^{\rm t}\Big)\boldsymbol{\nu}_0\Big]\Big|,$$

and to zero in $H^{1/2}(\Gamma^+_0)$.

As the lifting operator acting from $H^{1/2}(\gamma)$ in $H^1(O)$ is linear and continuous, the sequences $\tilde{\boldsymbol{W}}^1_\varepsilon$ and $\tilde{\boldsymbol{W}}^2_\varepsilon$ are converging, and

$$\|\tilde{\boldsymbol{W}}^1_\varepsilon - \tilde{\boldsymbol{W}}^2_\varepsilon\|_{H(\Omega_0)} \to 0$$

as $\varepsilon \to 0$. Hence, these sequences converge to one limit, which will be denoted by $\tilde{\boldsymbol{W}}_0$. The trace of the function $\tilde{\boldsymbol{W}}_0$ is $g_0\boldsymbol{\nu}_0$ on Γ_0^- and equals zero on Γ_0^+. It follows from here that

$$[\tilde{\boldsymbol{W}}_0]\boldsymbol{\nu}_0 = \Big|\Big[(\nabla w_0)^{\mathrm{t}}\Big(\frac{\partial \boldsymbol{V}}{\partial \boldsymbol{x}} + \Big(\frac{\partial \boldsymbol{V}}{\partial \boldsymbol{x}}\Big)^{\mathrm{t}}\Big)\boldsymbol{\nu}_0\Big]\Big| \qquad \text{almost everywhere on } \Gamma_0. \tag{25}$$

Thus, the following lemma is proved.

Lemma 2. *These exists a function $\tilde{\boldsymbol{W}}_0 \in H(\Omega_0)$ such that*

$$\tilde{\boldsymbol{W}}_\varepsilon^1 \to \tilde{\boldsymbol{W}}_0 \qquad \textit{strongly in } H^{1,0}(\Omega_0) \times H^{1,0}(\Omega_0),$$

$$\tilde{\boldsymbol{W}}_\varepsilon^2 \to \tilde{\boldsymbol{W}}_0 \qquad \textit{strongly in } H^{1,0}(\Omega_0) \times H^{1,0}(\Omega_0).$$

The function $\tilde{\boldsymbol{W}}_0$ satisfies Eq. (25).

Let us consider the functions $\boldsymbol{U}_\varepsilon^1$ and $\boldsymbol{U}_\varepsilon^2$ introduced in proving Lemma 1.

Lemma 3. *The functions $\boldsymbol{U}_\varepsilon^1$ and $\boldsymbol{U}_\varepsilon^2$ are characterized by the convergence*

$$\begin{aligned} \boldsymbol{U}_\varepsilon^1 &\to \boldsymbol{Q}_0 + \tilde{\boldsymbol{W}}_0 \qquad \textit{strongly in } H^{1,0}(\Omega_0) \times H^{1,0}(\Omega_0), \\ \boldsymbol{U}_\varepsilon^2 &\to -\boldsymbol{Q}_0 + \tilde{\boldsymbol{W}}_0 \qquad \textit{strongly in } H^{1,0}(\Omega_0) \times H^{1,0}(\Omega_0), \end{aligned} \tag{26}$$

where

$$\boldsymbol{Q}_0 = \boldsymbol{W}_0^{\mathrm{t}} \frac{\partial \boldsymbol{V}}{\partial \boldsymbol{x}}.$$

Proof. By virtue of Eq. (24), we can pass to the limit with $\varepsilon \to 0$ in Eq. (23). Then, we obtain the first convergence in (26). The second convergence in (26) is obvious. Lemma 3 is proved.

Thus, generalizing two last lemmas, we obtain the following convergence for $\varepsilon \to 0$:

$$\boldsymbol{\lambda}_\varepsilon^1 \to (\boldsymbol{Q}_0 + \tilde{\boldsymbol{W}}_0, 0) \qquad \text{strongly in } H(\Omega_0),$$

$$\boldsymbol{\lambda}_\varepsilon^2 \to (-\boldsymbol{Q}_0 + \tilde{\boldsymbol{W}}_0, 0) \qquad \text{strongly in } H(\Omega_0).$$

3. Obtaining the Energy Functional Derivative. Let us consider a function

$$\alpha(\varepsilon) = \frac{\Pi(\Omega_\varepsilon; \boldsymbol{\chi}^\varepsilon) - \Pi(\Omega_0; \boldsymbol{\chi}_0)}{\varepsilon},$$

where $\boldsymbol{\chi}_0$ and $\boldsymbol{\chi}^\varepsilon$ are solutions of problems (3) and (5), respectively.

Let us consider the functional $\Pi(\Omega_\varepsilon; \boldsymbol{\chi})$ of the potential energy of the solid occupying the perturbed domain Ω_ε. Let us apply the coordinate transformation (4) to integrals in $\Pi(\Omega_\varepsilon; \boldsymbol{\chi})$. As a result, we obtain the functional $\Pi_\varepsilon(\Omega_0; \boldsymbol{\chi})$, which can be written in the following form if Eqs. (16)–(18) are applied:

$$\Pi_\varepsilon(\Omega_0; \boldsymbol{\chi}) = \frac{1}{2}\int\limits_{\Omega_0} \sigma_{ij}(\boldsymbol{W})\varepsilon_{ij}(\boldsymbol{W})\,d\Omega_0 + \frac{1}{2}\int\limits_{\Omega_0} b(w,w)\,d\Omega_0 - \int\limits_{\Omega_0} \boldsymbol{f}^{\mathrm{t}}\boldsymbol{\chi}\,d\Omega_0$$

$$+\frac{1}{2}\,\varepsilon\int\limits_{\Omega_0}(A_1(\boldsymbol{V};\boldsymbol{W},\boldsymbol{W}) + A_2(\boldsymbol{V};w,w))\,d\Omega_0 - \varepsilon\int\limits_{\Omega_0}(\operatorname{div}(\boldsymbol{V}f_i)w_i + \operatorname{div}(\boldsymbol{V}f_3)w_3)\,d\Omega_0 + o(\varepsilon)R_4(\boldsymbol{\chi})$$

(R_4 is a certain bounded form). As the set $K_\varepsilon(\Omega_\varepsilon)$ is one-to-one mapped onto $K_\varepsilon(\Omega_0)$, we have

$$\Pi(\Omega_\varepsilon; \boldsymbol{\chi}^\varepsilon) = \Pi_\varepsilon(\Omega_0; \boldsymbol{\chi}_\varepsilon) \tag{27}$$

for all sufficiently small values of $\varepsilon > 0$.

Thus, by virtue of Eq. (27) and Lemma 1, we have

$$\frac{\Pi(\Omega_\varepsilon; \boldsymbol{\chi}^\varepsilon) - \Pi(\Omega_0; \boldsymbol{\chi}_0)}{\varepsilon} = \frac{\Pi_\varepsilon(\Omega_0; \boldsymbol{\chi}_\varepsilon) - \Pi(\Omega_0; \boldsymbol{\chi}_0)}{\varepsilon} \le \frac{\Pi_\varepsilon(\Omega_0; \boldsymbol{\chi}_0 + \varepsilon\boldsymbol{\lambda}_\varepsilon^1) - \Pi(\Omega_0; \boldsymbol{\chi}_0)}{\varepsilon}.$$

Hence, the following inequality is satisfied:

$$\limsup_{\varepsilon\to 0}\alpha(\varepsilon) \le \limsup_{\varepsilon\to 0}\frac{\Pi_\varepsilon(\Omega_0; \boldsymbol{\chi}_0 + \varepsilon\boldsymbol{\lambda}_\varepsilon^1) - \Pi(\Omega_0; \boldsymbol{\chi}_0)}{\varepsilon}. \tag{28}$$

Let us find the limit in the right side of Eq. (28). By virtue of Theorem 3, Lemmas 2 and 3, and boundedness of the form R_4, we obtain

$$\limsup_{\varepsilon\to 0}\frac{\Pi_\varepsilon(\Omega_0;\boldsymbol{\chi}_0+\varepsilon\boldsymbol{\lambda}^1_\varepsilon)-\Pi(\Omega_0;\boldsymbol{\chi}_0)}{\varepsilon}=\lim_{\varepsilon\to 0}\frac{\Pi_\varepsilon(\Omega_0;\boldsymbol{\chi}_0+\varepsilon\boldsymbol{\lambda}^1_\varepsilon)-\Pi(\Omega_0;\boldsymbol{\chi}_0)}{\varepsilon}$$

$$=\int\limits_{\Omega_0}(\sigma_{ij}(\boldsymbol{W}_0)\varepsilon_{ij}(\boldsymbol{Q}_0)-\bar{\boldsymbol{f}}^{\rm t}\boldsymbol{Q}_0)\,d\Omega_0+\int\limits_{\Omega_0}(\sigma_{ij}(\boldsymbol{W}_0)\varepsilon_{ij}(\tilde{\boldsymbol{W}}_0)-\bar{\boldsymbol{f}}^{\rm t}\tilde{\boldsymbol{W}}_0)\,d\Omega_0$$

$$+\frac{1}{2}\int\limits_{\Omega_0}A_1(\boldsymbol{V};\boldsymbol{W}_0,\boldsymbol{W}_0)\,d\Omega_0+\frac{1}{2}\int\limits_{\Omega_0}A_2(\boldsymbol{V};w_0,w_0)\,d\Omega_0$$

$$-\int\limits_{\Omega_0}\Big(\operatorname{div}(\boldsymbol{V}f_i)w_{0i}+\operatorname{div}(\boldsymbol{V}f_3)w_0\Big)\,d\Omega_0,$$

where $\bar{\boldsymbol{f}}=(f_1,f_2)$.

At the same time, the following relations are valid:

$$\frac{\Pi(\Omega_\varepsilon;\boldsymbol{\chi}^\varepsilon)-\Pi(\Omega_0;\boldsymbol{\chi}_0)}{\varepsilon}=\frac{\Pi_\varepsilon(\Omega_0;\boldsymbol{\chi}_\varepsilon)-\Pi(\Omega_0;\boldsymbol{\chi}_0)}{\varepsilon}\geq\frac{\Pi_\varepsilon(\Omega_0;\boldsymbol{\chi}_\varepsilon)-\Pi(\Omega_0;\boldsymbol{\chi}_\varepsilon+\varepsilon\boldsymbol{\lambda}^2_\varepsilon)}{\varepsilon},$$

and hence, the following inequality is satisfied:

$$\liminf_{\varepsilon\to 0}\alpha(\varepsilon)\geq\liminf_{\varepsilon\to 0}\frac{\Pi_\varepsilon(\Omega_0;\boldsymbol{\chi}_\varepsilon)-\Pi(\Omega_0;\boldsymbol{\chi}_\varepsilon+\varepsilon\boldsymbol{\lambda}^2_\varepsilon)}{\varepsilon}.$$

By virtue of Theorem 3, Lemmas 2 and 3, and boundedness of the form R_4, we obtain

$$\liminf_{\varepsilon\to 0}\frac{\Pi_\varepsilon(\Omega_0;\boldsymbol{\chi}_\varepsilon)-\Pi(\Omega_0;\boldsymbol{\chi}_\varepsilon+\varepsilon\boldsymbol{\lambda}^2_\varepsilon)}{\varepsilon}=\lim_{\varepsilon\to 0}\frac{\Pi_\varepsilon(\Omega_0;\boldsymbol{\chi}_\varepsilon)-\Pi(\Omega_0;\boldsymbol{\chi}_\varepsilon+\varepsilon\boldsymbol{\lambda}^2_\varepsilon)}{\varepsilon}$$

$$=\int\limits_{\Omega_0}(\sigma_{ij}(\boldsymbol{W}_0)\varepsilon_{ij}(\boldsymbol{Q}_0)-\bar{\boldsymbol{f}}^{\rm t}\boldsymbol{Q}_0)\,d\Omega_0-\int\limits_{\Omega_0}(\sigma_{ij}(\boldsymbol{W}_0)\varepsilon_{ij}(\tilde{\boldsymbol{W}}_0)-\bar{\boldsymbol{f}}^{\rm t}\tilde{\boldsymbol{W}}_0)\,d\Omega_0$$

$$+\frac{1}{2}\int\limits_{\Omega_0}A_1(\boldsymbol{V};\boldsymbol{W}_0,\boldsymbol{W}_0)\,d\Omega_0+\frac{1}{2}\int\limits_{\Omega_0}A_2(\boldsymbol{V};w_0,w_0)\,d\Omega_0-\int\limits_{\Omega_0}(\operatorname{div}(\boldsymbol{V}f_i)w_{0i}+\operatorname{div}(\boldsymbol{V}f_3)w_0)\,d\Omega_0.$$

Let us introduce the following notation:

$$\Delta(\boldsymbol{W}_0)=\int\limits_{\Omega_0}(\sigma_{ij}(\boldsymbol{W}_0)\varepsilon_{ij}(\tilde{\boldsymbol{W}}_0)-\boldsymbol{f}^{\rm t}\tilde{\boldsymbol{W}}_0)\,d\Omega_0.$$

We can easily see that the upper and lower limits of the function $\alpha(\varepsilon)$ for $\varepsilon\to 0$ coincide if $\Delta(\boldsymbol{W}_0)=0$. Hence, there is a limit $\lim\limits_{\varepsilon\to 0}\alpha(\varepsilon)$, which is responsible for the energy functional derivative with respect to the domain-perturbation parameter.

Let us consider the expression $\Delta(\boldsymbol{W}_0)$ in more detail, using the generalized Green's formula [3]. As a result, we obtain

$$\Delta(\boldsymbol{W}_0)=-\int\limits_{\Omega_0}(\sigma_{ij,j}(\boldsymbol{W}_0)+f_i)\tilde{w}_{0i}\,d\Omega_0-\langle\sigma_\nu(\boldsymbol{W}_0),[\tilde{\boldsymbol{W}}^{\rm t}_0]\boldsymbol{\nu}_0\rangle_{\Gamma_0}-\langle\sigma_{\tau i}(\boldsymbol{W}_0),[\tilde{w}_{0\tau i}]\rangle_{\Gamma_0},$$

where $\tilde{w}_{0\tau_0 i}=\tilde{w}_{0i}-(\tilde{\boldsymbol{W}}^{\rm t}_0\boldsymbol{\nu}_0)\nu_{0i}$ and $\boldsymbol{\sigma}_\tau(\boldsymbol{W}_0)$ is the shear stress vector.

The function $\boldsymbol{W}_0$ is known to satisfy the equilibrium equations

$$-\sigma_{ij,j}(\boldsymbol{W}_0)=f_i\quad(i=1,2)\qquad\text{almost everywhere in }\Omega_0$$

and a certain set of boundary conditions [3] for which

$$\sigma_\nu(\boldsymbol{W}_0)\leq 0,\qquad\boldsymbol{\sigma}_\tau(\boldsymbol{W}_0)=0\qquad\text{on }\Gamma_0.$$

The equilibrium equations are satisfies in the sense of distributions, and the boundary conditions on Γ_0 are satisfied in the sense of the space $H_{00}^{1/2}(\Gamma_0)$ and the space adjoint to the latter, $(H_{00}^{1/2}(\Gamma_0))^*$. As a result, we obtain

$$\Delta(\boldsymbol{W}_0) = -\Big\langle \sigma_\nu(\boldsymbol{W}_0), \Big|\Big[(\nabla w_0)^{\mathrm{t}}\Big(\frac{\partial \boldsymbol{V}}{\partial \boldsymbol{x}} + \Big(\frac{\partial \boldsymbol{V}}{\partial \boldsymbol{x}}\Big)^{\mathrm{t}}\Big)\boldsymbol{\nu}_0\Big]\Big|\Big\rangle_{\Gamma_0}.$$

Thus, Theorem 1 formulated in Sec. 1 is proved.

Finally, we should note that the equality of $\Delta(\boldsymbol{W}_0)$ to zero is only a sufficient condition of the existence of the energy functional derivative with respect to the domain-perturbation parameter, but not a necessary condition.

Let us give the following example. Let the solution $\boldsymbol{\chi}_0$ of the non-perturbed equilibrium problem be such that $[(\nabla w_0)^{\mathrm{t}}\boldsymbol{\nu}_0] \geq \delta$, where $\delta > 0$ is a certain constant. In addition, let the solution $\boldsymbol{\chi}_\varepsilon$ of problem (15) also satisfy the condition $[(\nabla w_\varepsilon)^{\mathrm{t}}\boldsymbol{\nu}_0] \geq \delta$. In this case, the following equality holds for all sufficiently small values of ε:

$$\Big|\Big[(\nabla w)^{\mathrm{t}}\boldsymbol{\nu}_0 - \varepsilon(\nabla w)^{\mathrm{t}}\Big(\frac{\partial \boldsymbol{V}}{\partial \boldsymbol{x}} + \Big(\frac{\partial \boldsymbol{V}}{\partial \boldsymbol{x}}\Big)^{\mathrm{t}} + \frac{r_{10}(\varepsilon)}{\varepsilon}\Big)\boldsymbol{\nu}_0\Big]\Big|$$
$$= [(\nabla w)^{\mathrm{t}}\boldsymbol{\nu}_0] - \varepsilon\Big[(\nabla w)^{\mathrm{t}}\Big(\frac{\partial \boldsymbol{V}}{\partial \boldsymbol{x}} + \Big(\frac{\partial \boldsymbol{V}}{\partial \boldsymbol{x}}\Big)^{\mathrm{t}} + \frac{r_{10}(\varepsilon)}{\varepsilon}\Big)\boldsymbol{\nu}_0\Big]$$

for $w = w_0$ and $w = w_\varepsilon$.

Let us consider the matrix equation

$$\boldsymbol{Y}^{\mathrm{t}} B = \boldsymbol{c}^{\mathrm{t}},$$

where

$$B = I - \varepsilon\Big(\frac{\partial \boldsymbol{V}}{\partial \boldsymbol{x}} - \frac{r_7(\varepsilon)}{\varepsilon}\Big), \qquad \boldsymbol{c}^{\mathrm{t}} = \boldsymbol{W}_0^{\mathrm{t}}\Big(\frac{\partial \boldsymbol{V}}{\partial \boldsymbol{x}} - \frac{r_7(\varepsilon)}{\varepsilon}\Big) - (\nabla w_0)^{\mathrm{t}}\Big(\frac{\partial \boldsymbol{V}}{\partial \boldsymbol{x}} + \Big(\frac{\partial \boldsymbol{V}}{\partial \boldsymbol{x}}\Big)^{\mathrm{t}} + \frac{r_{10}(\varepsilon)}{\varepsilon}\Big).$$

As in proving Lemma 1, we can show that the solution of such an equation exists (let us denote it by $\boldsymbol{Y}_\varepsilon^1$) and the function $\boldsymbol{\chi}_\varepsilon^1 = \boldsymbol{\chi}_0 + (\boldsymbol{Y}_\varepsilon^1, 0)$ belongs to $K_\varepsilon(\Omega_0)$.

Similarly, we have the function $\boldsymbol{\chi}_\varepsilon^2 = \boldsymbol{\chi}_\varepsilon - (\boldsymbol{Y}_\varepsilon^2, 0) \in K_0(\Omega_0)$, where

$$(\boldsymbol{Y}_\varepsilon^2)^{\mathrm{t}} = \boldsymbol{W}_\varepsilon^{\mathrm{t}}\Big(\frac{\partial \boldsymbol{V}}{\partial \boldsymbol{x}} - \frac{r_7(\varepsilon)}{\varepsilon}\Big) - (\nabla w_\varepsilon)^{\mathrm{t}}\Big(\frac{\partial \boldsymbol{V}}{\partial \boldsymbol{x}} + \Big(\frac{\partial \boldsymbol{V}}{\partial \boldsymbol{x}}\Big)^{\mathrm{t}} + \frac{r_{10}(\varepsilon)}{\varepsilon}\Big).$$

In addition, the following convergence is valid for the functions $\boldsymbol{Y}_\varepsilon^1$ and $\boldsymbol{Y}_\varepsilon^2$:

$$\boldsymbol{Y}_\varepsilon^1 \to \boldsymbol{Q}_0 + \boldsymbol{Y}_0 \qquad \text{strongly in } H^{1,0}(\Omega_0) \times H^{1,0}(\Omega_0),$$
$$\boldsymbol{Y}_\varepsilon^2 \to \boldsymbol{Q}_0 + \boldsymbol{Y}_0 \qquad \text{strongly in } H^{1,0}(\Omega_0) \times H^{1,0}(\Omega_0).$$

Here

$$\boldsymbol{Y}_0 = -(\nabla w_0)^{\mathrm{t}}\Big(\frac{\partial \boldsymbol{V}}{\partial \boldsymbol{x}} + \Big(\frac{\partial \boldsymbol{V}}{\partial \boldsymbol{x}}\Big)^{\mathrm{t}}\Big), \qquad \boldsymbol{Q}_0 = \boldsymbol{W}_0^{\mathrm{t}}\frac{\partial \boldsymbol{V}}{\partial \boldsymbol{x}}.$$

Based on the same reasoning as that used to prove Theorem 1, we can show that the upper and lower limits of the function $\alpha(\varepsilon)$ coincide as $\varepsilon \to 0$. Hence, the energy functional derivative exists and

$$\frac{d\Pi(\Omega_\varepsilon; \boldsymbol{\chi}^\varepsilon)}{d\varepsilon}\Big|_{\varepsilon=0} = \int\limits_{\Omega_0}\Big(\sigma_{ij}(\boldsymbol{W}_0)\varepsilon_{ij}(\boldsymbol{Q}_0) - \bar{\boldsymbol{f}}\boldsymbol{Q}_0\Big)\,d\Omega_0 + \int\limits_{\Omega_0}\Big(\sigma_{ij}(\boldsymbol{W}_0)\varepsilon_{ij}(\boldsymbol{Y}_0) - \bar{\boldsymbol{f}}\boldsymbol{Y}_0\Big)\,d\Omega_0$$
$$+ \frac{1}{2}\int\limits_{\Omega_0} A_1(\boldsymbol{V}; \boldsymbol{W}_0, \boldsymbol{W}_0)\,d\Omega_0 + \frac{1}{2}\int\limits_{\Omega_0} A_2(\boldsymbol{V}; w_0, w_0)\,d\Omega_0$$
$$- \int\limits_{\Omega_0}\Big(\operatorname{div}(\boldsymbol{V} f_i)w_{0i} + \operatorname{div}(\boldsymbol{V} f_3)w_0\Big)\,d\Omega_0. \tag{29}$$

Applying Green's formula to the second integral in the right side of Eq. (29) and taking into account the equilibrium equations and the boundary conditions on the crack, we obtain

$$\int\limits_{\Omega_0}\Big(\sigma_{ij}(\boldsymbol{W}_0)\varepsilon_{ij}(\boldsymbol{Y}_0) - \bar{\boldsymbol{f}}\boldsymbol{Y}_0\Big)\,d\Omega_0 = -\Big\langle \sigma_\nu(\boldsymbol{W}_0), \Big[(\nabla w_0)^{\mathrm{t}}\Big(\frac{\partial \boldsymbol{V}}{\partial \boldsymbol{x}} + \Big(\frac{\partial \boldsymbol{V}}{\partial \boldsymbol{x}}\Big)^{\mathrm{t}}\Big)\boldsymbol{\nu}_0\Big]\Big\rangle_{\Gamma_0}.$$

Let us denote the right side in Eq. (8) by G. Then, we obtain

$$\frac{d\Pi(\Omega_\varepsilon;\boldsymbol{\chi}^\varepsilon)}{d\varepsilon}\Big|_{\varepsilon=0} = G - \Big\langle \sigma_\nu(\boldsymbol{W}_0), [(\nabla w_0)^{\mathrm{t}}]\Big(\frac{\partial \boldsymbol{V}}{\partial \boldsymbol{x}} + \Big(\frac{\partial \boldsymbol{V}}{\partial \boldsymbol{x}}\Big)^{\mathrm{t}}\Big)\boldsymbol{\nu}_0\Big\rangle_{\Gamma_0}.$$

It does not seem possible to calculate the energy functional derivative with respect to the domain-perturbation parameter in the general case. Actually, the existence of such a derivative is not obvious, because the non-penetration condition contains the absolute value of the jump of the derivative over the normal to the function w_0, and the derivative may fail to exist.

Nevertheless, the proof of Theorem 1 implies that the following estimates are valid:

$$-\Delta(\boldsymbol{W}_0) = \liminf_{\varepsilon\to 0} \alpha(\varepsilon) - G \le \limsup_{\varepsilon\to 0} \alpha(\varepsilon) - G = \Delta(\boldsymbol{W}_0).$$

This work was supported by the Russian Foundation for Basic Research (Grant No. 06-01-00209) and by the Council on Grants of the President of the Russian Federation on the Support of Young Russian Scientists and Leading Scientific Schools (Grant No. MK-4338.2008.1).

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